Interaction of two liquid solitary waves in a trough resonator

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The interaction phenomena of two solitary waves observed by Wu, Keolian, and Rudnick [Phys. Rev. Lett. **52**, 1421 (1984)] in a trough resonator are analytically analyzed by the perturbative variation method. The coalescence period and the first coalescence time for same-polarity solitary waves as well as the interaction force and the interaction potential are given, which, to a certain extent, can explain the experimental phenomena well. The stable separation between two opposite-polarity solitary waves is explained to be the result of the balance of the soliton interaction force and an equivalent force due to the action of the two end walls on the solitons.

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I. INTRODUCTION

The discovery of a nonpropagating hydrodynamic soliton in a trough resonator was reported by Wu, Keolian, and Rudnick in 1984 [1,2]. Soon after, utilizing the multiple-scale method, Larraza and Putterman [3] obtained a single-solitary-wave solution. Similar results were obtained by Miles [4] by means of the averaged-Lagrangian method. Recently, the kink soliton on the surface of a liquid was observed by Denardo *et al.* [5]. In the experiment of Wu, Keolian, and Rudnick [1], they reported that "two solitons of the same polarity attract each other, but only weakly if the distance between them is significantly larger (say a factor of 3) than their effective length. Two solitons which start out 20 cm apart center-to-center, for example, take about 15 min to reach a separation where they strongly overlap .... A pair of solitons of opposite polarity in close proximity to each other repel each other and slowly move until they are approximately 12 cm apart, and then maintain this separation indefinitely." On this phenomenon, Ni and Wei [6] presented a theoretical analysis based on a new evolution equation which reduced from a new scale supposition. In this theory, the periodic coalescence is considered as the results of the fact that the wave form of two solitary waves is modulated by cosine factors (for the details, please see Ref. [6]). However, there exists a significant difference between the theoretical results and the experiments. Moreover, we find that the theory of soliton interaction based on the perturbative variational analysis (PVA) can better explain the phenomenon mentioned above. In this paper, based on the PVA presented by Bondeson, Anderson, and Lisak [7] and Anderson and Lisak [8], we obtained the coalescence period and the first coalescence time for a same-polarity solitary wave as well as the interaction force and potential existing between two same- and opposite-polarity solitons, which also provides a vivid particle description for the interaction between two nonpropagating solitary waves.

II. PERTURBATION VARIATIONAL SOLUTION FOR THE TWO-SOLITARY-WAVE INTERACTION

By means of a multiple scale method, Larraza and Putterman [5] obtained the evolution equation of a solitary wave in a trough resonator, that is,

$$i\omega\frac{\partial\varphi_1}{\partial t}-c^2\frac{\partial^2\varphi_1}{\partial x^2}+(\omega_1^2-\omega^2)\varphi_1-A|\varphi_1|^2\varphi_1=0\ , \quad (1)$$

where

$$c^2=g[T+kd(1-T^2)]/(2k)\ ,$$
$$A=k^4(6T^4-5T^2+16-9T^{-2})/8\ ,$$
$$\omega_1^2=kgT,\quad k=\pi/B,\quad T=\tanh(kd),\ g=980\ \mathrm{cm\,s^{-2}}\ ,$$

with B and d representing the depth and width of the trough, respectively. The relative height of the free surface from the static surface is

$$\zeta(x,y,t)=\frac{1}{g}\{[-i\omega\varphi_1e^{i\omega t}\cos(ky)+\mathrm{c.c.}]+[\tfrac{1}{2}k^2|\varphi_1|^2(T^2+1)\cos(2ky)+\tfrac{1}{2}k^2|\varphi_1|^2(T^2-1)]-[\tfrac{1}{4}k^2(3/T^2-1)\varphi_1^2e^{i2\omega t}\cos(2ky)+\mathrm{c.c.}]\}\ . \quad (2)$$

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Making the transformation

$$\tau=-(\omega_1^2-\omega^2)t/(2\omega) ,$$
$$\xi=\sqrt{(\omega_1^2-\omega^2)/c^2}x ,\quad (3)$$
$$\varphi_1=\sqrt{2(\omega_1^2-\omega^2)/A}\,\Psi$$

on Eq. (1), one obtains

$$i\frac{\partial\Psi}{\partial\tau}-\Psi+\frac{\partial^2\Psi}{\partial\xi^2}+2|\Psi|^2\Psi=0 . \quad (4)$$

Equation (4) gives a single propagating soliton solution,

$$\Psi(\xi,t)=2a\,\mathrm{sech}[2a(\xi-v\tau)]\times\exp[iv\xi/2-i\{(1-4a^2)v^2/4\}\tau] . \quad (5)$$

When $a=\frac{1}{2}$ and $v=0$, solution (5) is just the single nonpropagating solitary wave.

According to the method of Anderson and Lisak [8], the two-soliton solution can be approximated as a linear superposition of two well-separated single solitons, i.e.,

$$\Psi=\Psi_1+\Psi_2 , \quad (6)$$

$$\Psi_k=2a_k\mathrm{sech}[2a_k(\xi-\xi_k)]\times\exp[i2b_k(\xi-\xi_k)+i\varphi_k-i\tau]\quad (k=1,2) , \quad (7)$$

where $a_k(\tau)$, $\xi_k(\tau)$, $b_k(\tau)$, and $\varphi_k(\tau)$ are slowly varying functions of time. Since they are determined by the coupled nonlinear Schrödinger (NLS) equations given below, Eq. (6) is not the simple linear superposition under traditional meaning. It is clear that, when

$$2a_k=2a,\quad 2b_k=v/2 ,$$
$$-2b_k\xi_k(\tau)+\varphi_k(\tau)-\tau=-\{(1-4a^2)+v^2/4\}\tau ,$$

Eq. (7) becomes Eq. (5), which corresponds to the distantly separated case. When two solitons become slightly overlapped, the corresponding coupled NLS equations describing their interaction can be written as

$$i\frac{\partial\Psi_k}{\partial\tau}-\Psi_k+\frac{\partial^2\Psi_k}{\partial\xi^2}+2|\Psi_k|^2\Psi_k=-2T_{lk} ,$$
$$T_{lk}=\Psi_l^*\Psi_k^2+2\Psi_l|\Psi_k|^2 , \quad (8)$$

where $l\neq k$, $l,k=1,2$, and T_{lk} can be treated as a small perturbation. In order to investigate the slow change in soliton parameters that is due to their interaction, we reformulate Eqs. (8) as a variational principle for the soliton parameters according to the scheme of Refs. [7,8]. That is, Eqs. (8) can be derived from the Lagrangian

$$L=\int\int(L_I+L_c)d\xi\,d\tau , \quad (9)$$

where

$$L_I=\sum_{k=1,2}\left[\frac{1}{2}i\left[\Psi_k^*\frac{\partial\Psi_k}{\partial\tau}-\Psi_k\frac{\partial\Psi_k^*}{\partial\tau}\right]-|\Psi_k|^2-\left|\frac{\partial\Psi_k}{\partial\xi}\right|^2+|\Psi_k|^4\right] , \quad (10)$$

$$L_c=2(\Psi_1\Psi_2^*+\Psi_1^*\Psi_2)(|\Psi_1|^2+|\Psi_2|^2) , \quad (11)$$

L_I is the Lagrangian of the unperturbed NLS equation, and the Ψ_l dependence in T_{lk} is neglected in deriving L_c by the variational derivative. For simplicity and solvability, it is assumed that

$$|a_1-a_2|\ll a,\quad |b_1-b_2|\ll b ,$$
$$|a_1-a_2|\Delta\ll 1,\quad a\Delta\gg 1 , \quad (12)$$
$$|b_1-b_2|\Delta\ll 1 ,$$

where

$$a=\tfrac{1}{2}(a_1+a_2),\quad b=\tfrac{1}{2}(b_1+b_2) ,$$
$$\Delta=\xi_1-\xi_2,\quad \Delta\varphi=\varphi_2-\varphi_1 , \quad (13)$$
$$\beta=2b\Delta+\Delta\varphi .$$

Substituting Eq. (7) into Eqs. (9)–(11) results in

$$L=\int[\langle L_I\rangle+\langle L_c\rangle]d\tau , \quad (14)$$

where

$$\langle L_I\rangle=\int L_I d\xi=\sum_{k=1,2}\left[-4a_k\frac{d\varphi_k}{d\tau}+8a_kb_k\frac{d\xi_k}{d\tau}-16b_k^2a_k+\tfrac{16}{3}a_k^3\right] , \quad (15)$$

$$\langle L_c\rangle=\int L_c d\xi=4(16)\int[a_1^3a_2\mathrm{sech}^3(Z_1)\mathrm{sech}Z_2+a_2^3a_1\mathrm{sech}^3(Z_2)\mathrm{sech}Z_1]\cos[\Delta\varphi+2b_1\xi_1-2b_2\xi_2-2(b_2-b_1)]d\xi . \quad (16)$$

Taking the variational derivative with respect to the parameters leads to

$$\frac{\delta L}{\delta\varphi_k}=0\to 4\frac{da_k}{d\tau}=-\frac{\partial\langle L_c\rangle}{\partial\varphi_k} , \quad (17)$$

$$\frac{\delta L}{\delta\xi_k}=0\to 8\frac{d(a_kb_k)}{d\tau}=\frac{\partial\langle L_c\rangle}{\partial\xi_k} , \quad (18)$$

$$\frac{\partial L}{\delta b_k}=0\to 8a_k\frac{d\xi_k}{d\tau}-32a_kb_k=-\frac{\partial\langle L_c\rangle}{\partial b_k} , \quad (19)$$

$$\frac{\delta L}{\delta a_k}=0\rightarrow 8b_k\frac{d\xi_k}{d\tau}-4\frac{d\varphi_k}{d\tau}-16b_k^2+16a_k^2=-\frac{\partial\langle L_c\rangle}{\partial a_k} . \tag{20}$$

By using approximation (12) and neglecting the relatively smaller quantities in the integral, one obtains

$$\langle L_c\rangle=4(64)a^3\cos(\beta)e^{-2a\Delta} . \tag{21}$$

In the calculation of $\partial\langle L_c\rangle/\partial a_k$ and $\partial\langle L_c\rangle/\partial a_k$, a term in which a factor $(\xi-\xi_k)$ is included and appears after the derivative with respect to b_k should be considered. For example,

$$\frac{\partial\langle L_c\rangle}{\partial b_1}=4(32)a^4\sin(\beta)I ,$$
$$I=\int_{-\infty}^{+\infty}\left\{\frac{8(2)}{[e^{2a_1(\xi-\xi_1)}+e^{-2a_1(\xi-\xi_1)}]^3[e^{2a_2(\xi-\xi_2)}+e^{-2a_2(\xi-\xi_2)}]}+\frac{8(2)}{[e^{2a_1(\xi-\xi_1)}+e^{-2a_1(\xi-\xi_1)}][e^{2a_2(\xi-\xi_2)}+e^{-2a_2(\xi-\xi_2)}]^3}\right\}(\xi-\xi_1)d\xi . \tag{22}$$

According to Eqs. (12), the first term in the above integral is far smaller than the second term. The following approximation can be used:

$$I\approx\int_{-\infty}^{-(1/2)(\xi_1-\xi_2)}e^{6a\xi_2-6a\xi_1}\frac{32z\,dz}{(e^{2az}+e^{-2az}C_1^2)^3e^{-2ax}}\approx\frac{1}{2a^2}e^{-2a\Delta}-\frac{16\Delta}{8a}e^{-2a\Delta} , \tag{23}$$

with $z=\xi-\xi_1$ and $C_1^2=e^{-4a\Delta}$. Similarly, $-\partial\langle L_c\rangle/\partial b_2$ and $\partial\langle L_c\rangle/\partial a_k$ can be calculated. Substituting Eqs. (21) and (22) into Eqs. (17)–(20), and combining them, results in

$$\frac{d(ia_k+b_k)}{d\tau}=(-1)^k 64a^3e^{i\beta-2a\Delta} , \tag{24}$$

$$\frac{d\xi_k}{d\tau}=4b_k-8a\sin(\beta)e^{-2a\Delta}+32a^2\Delta\sin(\beta)e^{-2a\Delta} , \tag{25}$$

$$\frac{d\varphi_k}{d\tau}=4(b_k^2+a_k^2)-16ab\sin(\beta)e^{-2a\Delta}+64a^2b\Delta\sin(\beta)e^{-2a\Delta}+96a^2\sin(\beta)e^{-2a\Delta}-64a^3\Delta\cos(\beta)e^{-2a\Delta} . \tag{26}$$

The second term in Eq. (25) and the third and fifth terms in Eq. (26) are omitted in Ref. [9]. Besides, comparing the results with Ref. [9], one can see that the discrepancies in Eqs. (25) and (26) also result from the sign discrepancy of the first term in Eq. (23). However, those differences do not influence the discrepancies $\xi_1-\xi_2$ and $\varphi_2-\varphi_1$, though they have an effect on the quantities ξ_k and φ_k themselves. Combining the two equations in Eq. (24), one obtains $i(a_1+a_2)+(b_1+b_2)=\text{const}$; that means

$$2a=a_1+a_2=\text{const} , \tag{27}$$

$$2b=b_1+b_2=\text{const} . \tag{28}$$

By using Eqs. (27) and (28), one obtains

$$\frac{d\Delta}{d\tau}=-4(b_2-b_1) , \tag{29}$$

$$\frac{d\Delta\varphi}{d\tau}=8a(a_2-a_1)+8b(b_2-b_1) \tag{30}$$

from Eqs. (25) and (26), respectively. By making derivatives on $X=Na^2e^{i\beta-2a\Delta}$ and using Eqs. (24), (29), and (30), an equation of $Y=i(a_1-a_2)+(b_2-b_1)$ is derived,

$$Y^2-32a^2e^{i\beta-2a\Delta}=M^2=\text{const} , \tag{31}$$

where $N=32$ is obtained by comparing the coefficients. Combining Eqs. (24) and (31) results in

$$\frac{dY}{d\tau}=4a[Y^2-M^2] , \tag{32}$$

which has a solution

$$Y=-M\tanh[4Ma\tau+\alpha_1+i\alpha_2] . \tag{33}$$

Substituting $(b_2-b_1)=\text{Re}(Y)$ into Eq. (29), one easily obtains

$$\Delta(\tau)-\Delta(0)=\frac{1}{2a}\ln\frac{\cosh[8ap\tau+2\alpha_1]+\cos[8aq\tau+2\alpha_2]}{\cosh(2\alpha_1)+\cos(2\alpha_2)} , \tag{34}$$

with

$$iq+p=M=i4\sqrt{2}ae^{i\beta/2-a\Delta} . \tag{35}$$

Considering the symmetry in the experiment, the amplitude and the velocity of the two solitons are equal, i.e., $a_1=a_2$ and $b_1=b_2$. When the solitons are at rest at the beginning, i.e., $b_1=b_2=0$ or $b=0$, Eqs. (34) and (35) become (set $\alpha_1=\alpha_2=0$)

$$\Delta(\tau)-\Delta(0)=\frac{1}{2a}\ln\frac{\cosh[8ap\tau]+\cos[8aq\tau]}{2} , \tag{36}$$

$$iq+p=i4\sqrt{2}ae^{i\Delta\varphi(0)/2-a\Delta(0)} , \tag{37}$$

where $\Delta(0)=\xi_1(0)-\xi_2(0)$, $\Delta\varphi(0)=\varphi_2(0)-\varphi_1(0)$. Equation (36) represents the time evolution of two solitons when they are interacting.

III. INTERACTION OF TWO SAME-POLARITY SOLITONS

For the same-polarity solitons, where the initial phase discrepancy $\Delta\varphi(0)=0$, Eq. (36) reduces to

$$\Delta(\tau)-\Delta(0)=\frac{1}{a}\ln[\cos(16\sqrt{2}a^2e^{-a\Delta(0)}\tau)] , \quad (38)$$

which represents the periodic coalescence of solitons observed in the experiment of Wu, Keolian, and Rudnick [1]. The period is

$$T_c=\frac{\pi\exp[a\Delta(0)]}{8\sqrt{2}a^2} . \quad (39)$$

It will take the time

$$\tau_1=\frac{1}{\sqrt{2}16a^2}e^{a\Delta(0)}\arccos[e^{-a\Delta(0)}] \quad (40)$$

for solitons to move from the initial separation $\Delta(0)$ to the first coalescence $[\Delta(\tau)=0]$. Obviously, the period T_c and the time τ_1 are very sensitive to the amplitude and initial separation. Unfortunately, it is difficult to determine the soliton amplitude according to the description in Ref. [1]. In addition, at present, the theoretical determination of the amplitude is not precisely identical to the experimental one. According to Ref. [1], the experimental expression of the shape for the one-soliton solution is

$$\zeta(x,y,t)=[2.8e^{-1.1y}-0.70]\text{sech}(x/1.12) \times\cos[2\pi(5.1)t] \text{ cm} . \quad (41)$$

Compared with Eq. (2), the theoretical result, by setting $f=5.1$ Hz, there exists some discrepancy between the experiment and theory. Hence the initial amplitude must be estimated. Comparing Eqs. (2) and (41), at $y=0$, the amplitude of $\zeta(x,y,t)$ in Eq. (2) can be approximated by A_1+A_2, with A_1 and A_2 representing the amplitudes of the (0,1) mode and the (0,2) mode, respectively. At $y=B$, it can be approximated by A_1-A_2. On the other hand, the values of the $\zeta(x,y,t)$ at $y=0$ and $y=B$ are 2.1 and 0.53 cm, respectively, according to Eq. (41), which results in $A_1=1.32$ and $A_2=0.78$. Finally, $a\approx0.82$ can be determined by using Eq. (2) and the scale transformation (3). Substituting this into Eq. (40) as the initial amplitude and setting the initial separation to be 20 cm according to the experiment [1], the time for the first coalescence can be obtained, i.e., $t_1=21.8$ min for $f=5.1$ Hz and $t_1=7.6$ and 2.3 min for $f=5.2$ and 5.32 Hz, respectively. The experimental result (about 15 min) is included in the above values. Note that above value of the amplitude $(a=0.82)$ is determined from Eq. (41) obtained by $f=5.1$ Hz. It must be very different from 0.82 in the case of $f=5.2$ Hz and $f=5.32$ Hz. Therefore the perturbation variational analysis presented in this paper can explain the experiment fairly well.

It should be pointed out that Eq. (38) is not suitable for the case of a short distance, since it is based on supposition (12). One may also find some uncertainty in determining the coalescence time τ_1 with Eq. (40), which is derived from Eq. (38). However, the time τ_1 can be considered as two parts. One is taken at the distant separation, the other is taken at the small separation. At the distant separation, Eq. (12) is quite suitable because $a\Delta\gg1$ is satisfied; since two solitons approach very slowly, it will take a very long time. With Δ decreasing, two solitons attract in an increasing manner and it takes a very short time at the small separation. Therefore the actual values of the period T_c and coalescence time τ_1 can be represented by the approximation determined by Eqs. (39) and (40).

IV. INTERACTION OF TWO OPPOSITE-POLARITY SOLITARY WAVES AND THE TWO-NONPROPAGATING-SOLITONS STATE

For the opposite-polarity waves, the initial phase discrepancy $\Delta\varphi(0)=\pi$; then

$$\Delta(\tau)-\Delta(0)=\frac{1}{a}\ln[\cosh(16\sqrt{2}a^2e^{-a\Delta(0)}\tau)] , \quad (42)$$

which shows that two solitons are moving apart uniformly. However, this phenomenon is not observed and cannot be observed. Wu, Keolian, and Rudnick [1] observed that the solitons repel each other and slowly move until they are approximately 12 cm apart, and then maintain this separation indefinitely. We consider that this stability is due to the pinning effects of the two end walls, which is equivalent to a force on the solitons. The equivalent force balances the repulsion force between the two solitons, and then the two solitons maintain their separation indefinitely, which can be regarded as a new stable mode, called the two-nonpropagating-solitons state. The equivalent force of one of the end walls can be estimated by the repulsion force of the two solitons because the two forces should be equal when stable. One can find the interaction force to be

$$F=\frac{1}{2}\frac{d^2\Delta(\tau)}{d\tau^2}=\pm256a^3e^{-2a\Delta} , \quad (43)$$

where + represents repulsion from Eq. (42) and − represents attraction from Eq. (38). The equivalent force of the wall can be calculated by substituting the stable separation into Eq. (43). Let $r=\frac{1}{2}\Delta$, i.e., the distance of each soliton from the center of the trough; the interaction potential can be introduced as

$$U(r)=-\int_0^r F(\Delta)dr=\mp64a^2[1-e^{-4ar}] . \quad (44)$$

This is the repulsion (−) and attraction (+) potential provided by each soliton for the others.

In summary, the interaction phenomena of two solitary waves observed by Wu, Keolian, and Rudnick [1] in a trough resonator are analyzed by the perturbative variation method. The coalescence period and first coalescence time for same-polarity solitons as well as the interaction force and the interaction potential are given, which can explain the experimental phenomena to a certain extent. It is hypothesized in this paper that the two end walls have an effect on the opposite-polarity solitons. Finally, the solitons maintain their separation indefinitely and form a stable mode called the two-nonpropagating-solitons state, and, at the stable state, the force on the solitary wave provided by the walls balances the repulsion interaction force of the two solitons.

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